A PRIORI BOUNDS AND A LIOUVILLE THEOREM ON A HALF-SPACE FOR HIGHER-ORDER ELLIPTIC DIRICHLET PROBLEMS

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Abstract. We consider the $2m$-th order elliptic boundary value problem $Lu = f(x,u)$ on a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary conditions $u = \frac{\partial}{\partial\nu}u = \ldots = (\frac{\partial}{\partial\nu})^{m-1}u = 0$ on $\partial\Omega$. The operator L is a uniformly elliptic operator of order $2m$ given by $L = \big(-\sum_{i,j=1}^N a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\big)^m + \sum_{|\alpha|\leq 2m-1} b_\alpha(x)D^\alpha$. For the nonlinearity we assume that $\lim_{s\to\infty}\frac{f(x,s)}{s^q} = h(x)$, $\lim_{s\to-\infty}\frac{f(x,s)}{|s|^q} = k(x)$ where $h, k \in C(\overline{\Omega})$ are positive functions and $q > 1$ if $N \leq 2m$, $1 < q < \frac{N+2m}{N-2m}$ if $N > 2m$. We prove a priori bounds, i.e, we show that $\|u\|_{L^\infty(\Omega)} \leq C$ for every solution u, where $C > 0$ is a constant. The solutions are allowed to be sign-changing. The proof is done by a blow-up argument which relies on the following new Liouville-type theorem on a half-space: if u is a classical, bounded, non-negative solution of $(-\Delta)^m u = u^q$ in $\mathbb{R}^N_+$ with Dirichlet boundary conditions on $\partial\mathbb{R}^N_+$ and $q > 1$ if $N \leq 2m$, $1 < q \leq \frac{N+2m}{N-2m}$ if $N > 2m$ then $u \equiv 0$.

1. Introduction

A priori bounds for solutions of elliptic boundary value problems have been of major importance at least as far back as Schauder's work in the 1930s. In this paper we prove a priori estimates on bounded, smooth domains $\Omega \subset \mathbb{R}^N$ for solutions of higher order boundary value problems of the form

$$Lu = f(x,u) \text{ in } \Omega, \quad u = \frac{\partial}{\partial\nu}u = \ldots = \left(\frac{\partial}{\partial\nu}\right)^{m-1} u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Here ν is the unit exterior normal on $\partial\Omega$ and

$$L = \Big(-\sum_{i,j=1}^N a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\Big)^m + \sum_{|\alpha|\leq 2m-1} b_\alpha(x)D^\alpha$$

is a uniformly elliptic operator with coefficients $b_\alpha \in L^\infty(\Omega)$ and $a_{ij} \in C^{2m-2}(\overline{\Omega})$ such that there exists a constant $\lambda > 0$ with $\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^N$, $x \in \Omega$. Our main result is the following:

Theorem 1. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\partial\Omega \in C^{2m}$. Let $m \in \mathbb{N}$ and assume that $q > 1$ if $N \leq 2m$ and $1 < q < \frac{N+2m}{N-2m}$ if $N > 2m$. Suppose further that there exist*

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positive, continuous functions $k, h : \overline{\Omega} \to (0, \infty)$ *such that*

$$\lim_{s \to +\infty} \frac{f(x,s)}{s^q} = h(x), \quad \lim_{s \to -\infty} \frac{f(x,s)}{|s|^q} = k(x) \tag{1.2}$$

uniformly with respect to $x \in \overline{\Omega}$. *Then there exists a constant* $C > 0$ *depending only on the data* $a_{ij}, b_\alpha, \Omega, N, q, h, k$ *such that* $\|u\|_\infty \leq C$ *for every solution* u *of* (1.1).

Remark 2. *Suppose the nonlinearity depends on a real parameter* λ, *i.e,* $f_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ *and*

$$\lim_{s \to +\infty} \frac{f_\lambda(x,s)}{\lambda s^q} = h(x), \quad \lim_{s \to -\infty} \frac{f_\lambda(x,s)}{\lambda |s|^q} = k(x)$$

uniformly with respect to $x \in \Omega$ *and* $\lambda \in [\lambda_0, \infty)$ *where* $\lambda_0 > 0$. *Then the a priori bound of Theorem 1 depends additionally on* λ_0 *but not on* λ. *This is important in the study of global solution branches of a parameter dependent version of* (1.1), *which we will pursue in future work.*

We are focusing on the case of superlinear nonlinearities $f(x,u)$ with subcritical growth. A model nonlinearity is $f(x,s) = |s|^q$. Our results hold with no restriction on the shape of the domain Ω and for general, possibly sign-changing solutions. This is important since the lack of the maximum principle for higher order equations does not allow to restrict attention to positive solutions only.

In the second-order case $m = 1$ a priori bounds for positive solutions have been established for subcritical, superlinear nonlinearities via different methods by Brezis, Turner [7], Gidas, Spruck [13], DeFigueiredo, Lions and Nussbaum [10] and recently by Quittner, Souplet [20] and McKenna, Reichel [19]. In the higher-order case $m \geq 2$ the theory is far less developed and strongly depends on the type of boundary conditions considered. For Dirichlet boundary conditions we only know of a result of Soranzo [23], who proved a priori bounds for positive radial solutions on a ball if $L = (-\Delta)^m$. For Navier boundary conditions the picture is more complete. Let $L = (-L_0)^m$ where $L_0 = a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + b_\alpha \frac{\partial}{\partial x_\alpha}$ is a second order operator and suppose the boundary conditions are of Navier-type:

$$u = (-L)u = \ldots = (-L)^{m-1} u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

Soranzo [23] proved a priori bounds for positive solutions if $L_0 = \Delta$ and Ω is a bounded smooth convex domain. Recently, Sirakov [22] improved this result to general operators $L = (-L_0)^m$ and general bounded smooth domains. Both authors strongly use the fact that the boundary conditions (1.3) allow to write the problem as a coupled system of second order equations, where each equation is complemented with Dirichlet boundary conditions. In this case maximum principles are available. In contrast, the higher order Dirichlet problem can not be rewritten as a system and therefore requires different techniques.

In our approach we extend the so-called "scaling argument" of Gidas and Spruck [13], which they used to deal with the second order case $m = 1$ and positive solutions. Let us give a brief sketch of their method. Gidas and Spruck assume that there exists a sequence of positive solutions with L^∞-norm tending to $+\infty$. After rescaling the solutions to norm 1 and blowing-up the coordinates one can take a limit of the rescaled solutions and obtains

a nontrivial positive solution of a limit boundary value problem $-\Delta u = u^q$ on either the full-space $\mathbb{R}^N$ or the half-space $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_1 > 0\}$ together with Dirichlet boundary conditions. Then a contradiction is reached provided that a Liouville-type result is available, i.e., a result which shows that the non-negative solutions of the limit problem must be identically zero. For subcritical q Gidas and Spruck [13], [14] proved both the full-space and the half-space Liouville theorem for $-\Delta u = u^q$ via the method of moving planes.

In order to deal with the higher order Dirichlet problem (1.1) and solutions which may change sign, the blow up procedure has to be modified. Indeed, even under assumption (1.2), there seems to be no direct argument to exclude the case of *negative blow up* (i.e., the existence of a sequence of solutions which is not uniformly bounded from below). Instead, it is excluded a posteriori after passing to the limit equation. Once this is done, we still need Liouville theorems for nonnegative solutions of the higher order problems on $\mathbb{R}^N$, $\mathbb{R}^N_+$. The full-space Liouville theorem stated next is already known; it was proved by Lin [18] if $m = 2$ and for general $m \geq 2$ by Wei, Xu [24].

Theorem 3 (Wei, Xu). *Let $m \in \mathbb{N}$ and assume that $q > 1$ if $N \leq 2m$ and $1 < q < \frac{N+2m}{N-2m}$ if $N > 2m$. If u is a classical non-negative solution of*

$$(-\Delta)^m u = u^q \ \text{ in } \mathbb{R}^N,$$

then $u \equiv 0$.

Even in the case of the Navier boundary conditions, the corresponding Liouville theorem for the polyharmonic problem in the half-space is harder to achieve and has only recently been proved by Sirakov [22]. Due to the lack of a (local) maximum principle, the corresponding Dirichlet problem is even more difficult to deal with. Here we show the following new Liouville Theorem for the half-space which complements Theorem 3.

Theorem 4. *Let $m \in \mathbb{N}$ and assume that $q > 1$ if $N \leq 2m$ and $1 < q \leq \frac{N+2m}{N-2m}$ if $N > 2m$. If u is a classical non-negative bounded solution of*

$$(-\Delta)^m u = u^q \ \text{ in } \mathbb{R}^N_+, \quad u = \frac{\partial}{\partial x_1} u = \ldots = \frac{\partial^{m-1}}{\partial x_1^{m-1}} u = 0 \ \text{ on } \partial\mathbb{R}^N_+ \tag{1.4}$$

then $u \equiv 0$.

We point out that the critical case $q = \frac{N+2m}{N-2m}$ is allowed in Theorem 4. Note also that Theorem 4 holds in the class of bounded solutions. It remains an open problem to extend the result to the class of all (possibly unbounded) classical positive solutions.

Let us outline the proof of Theorem 4 and point out the main difficulties. Following Gidas and Spruck, we transform the half-space problem via a Kelvin inversion into a problem in the unit ball, where the point at infinity is mapped onto the boundary point $P = (-1, 0, \ldots, 0)$. The transformed solution satisfies

$$\begin{aligned} &(-\Delta)^m v = 2^{2m} |x - P|^{-\alpha} v^q && \text{pointwise in } B_1(0), \\ &v = \frac{\partial}{\partial \nu} v = \ldots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} v = 0 && \text{on } \partial B_1(0) \setminus \{P\}, \end{aligned} \tag{1.5}$$

where $\alpha = N+2m-q(N-2m) \geq 0$. The key step is to show that v is axially symmetric around the x_1-axis. In the second order case $m = 1$ this is proved with the classical moving plane method, which is a local method based on the maximum principle. The same local approach fails for the higher order case $m \geq 2$ since the maximum principle is not available. Very recently, a new development in the moving plane procedure by Berchio, Gazzola and Weth [4] overcame part of this difficulty. The authors made the moving plane method applicable for classical solutions of polyharmonic Dirichlet problems on balls. Instead of a local maximum principle method they argue via the Green integral-representation and properties of the Green function. However, the method of [4] does not apply here, since the solution v of (1.5) may have a singularity at $P \in \partial B_1(0)$. To overcome this problem, a large part of this work is devoted to show that every solution v of (1.5) which corresponds to a bounded solution of (1.4) can be of represented via the Green function. In this step we also use Green function estimates of Grunau and Sweers [15]. Then we apply a moving plane argument, using the Green function representation and the Hardy-Littlewood-Sobolev inequality, to get the desired symmetry result. Comparing this variant of the moving plane method with the one in [4], we point out that Berchio, Gazzola and Weth allow more general (non-Lipschitz) nonlinearities, but their argument relies on Green function representations for directional derivatives of the solution which in our situation might not exist.

Once the symmetry result for v is established, we readily conclude – following Gidas and Spruck [13], [14] again – that the corresponding solution u of (1.4) is axially symmetric around *any* axis parallel to the x_1-axis. Consequently, u is a function of x_1 only and hence solves an ordinary differential equation. It is then easy to conclude that $u \equiv 0$.

We recall that the original moving plane method goes back to Alexandrov [3] and Serrin [21] and was further developed by Gidas, Ni, Nirenberg [12] for second order equations. Recent improvements of the moving plane method for higher order equations and pseudo differential operators using integral representations rather than local maximum principles were achieved by Chang, Yang [8], Berchio, Gazzola, Weth [4], Li [16], Chen, Li, Ou [9] and Birkner, López-Mimbela, Wakolbinger [5].

The paper is organized as follows. In Section 2 we prove Theorem 1 assuming Theorem 4. We give the details of the blow-up procedure taking into account that we allow for solutions blowing up to either at $+\infty$ or $-\infty$. The rest of the paper is devoted to the proof of Theorem 4. In Section 3 we prove a Green-representation formula on half-spaces (Theorem 9) by approximating the half-space by a family of growing balls. Based on the Green-representation for balls and by a careful estimate of the boundary integrals and the monotone convergence theorem we obtain a Green-representation for the half-space. Finally, in Section 4 we prove Theorem 4.

2. Proof of Theorem 1 – the blow-up argument

In this section we give the details of the blow-up argument for the proof of Theorem 1 under the assumption of the validity of the Liouville-type result of Theorem 4. The proof

uses standard linear L^p-$W^{2m,p}$ estimates for linear problems

$$Lu = g(x) \text{ in } \Omega, \tag{2.1}$$

$$u = \frac{\partial}{\partial \nu} u = \ldots = \left(\frac{\partial}{\partial \nu}\right)^{m-1} u = 0 \text{ on } \partial\Omega. \tag{2.2}$$

Recall the following basic estimate of Agmon, Douglis, Nirenberg [2].

Theorem 5 (Agmon, Douglis, Nirenberg)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\partial\Omega \in C^{2m}, m \in \mathbb{N}$. Let $a_{ij} \in C^{2m-2}(\overline{\Omega})$, $b_\alpha \in L^\infty(\Omega)$, $g \in L^p(\Omega)$ for some $p \in (1,\infty)$. Suppose $u \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ satisfies (2.1). Then there exists a constant $C > 0$ depending only on $\|a_{ij}\|_{C^{2m-2}}, \|b_\alpha\|_\infty, \lambda, \Omega, N, p, m$ and the modulus of continuity of a_{ij} such that*

$$\|u\|_{W^{2m,p}(\Omega)} \le C(\|g\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}).$$

We will also be using the following local analogue of this result. Though the proof may be standard we give it for the reader's convenience.

Corollary 6. *Let Ω be a ball $\{x \in \mathbb{R}^N : |x| < R\}$ or a half-ball $\{x \in \mathbb{R}^N : |x| < R, x_1 > 0\}$. Let $m \in \mathbb{N}$, $a_{ij} \in C^{2m-2}(\overline{\Omega})$, $b_\alpha \in L^\infty(\Omega)$, $g \in L^p(\Omega)$ for some $p \in (1,\infty)$. Suppose $u \in W^{2m,p}(\Omega)$ satisfies (2.1)*

(i) *either on the ball*
(ii) *or on the half-ball together with the boundary conditions $u = \frac{\partial}{\partial x_1} u = \ldots = \frac{\partial^{m-1}}{\partial x_1^{m-1}} u = 0$ on $\{x \in \mathbb{R}^N : |x| < R, x_1 = 0\}$.*

Then there exists a constant $C > 0$ depending only on $\|a_{ij}\|_{C^{2m-2}}, \|b_\alpha\|_\infty, \lambda, \Omega, N, p, m,$ the modulus of continuity of a_{ij} and R such that for any $\sigma \in (0,1)$

$$\|u\|_{W^{2m,p}(\Omega \cap B_{\sigma R})} \le \frac{C}{(1-\sigma)^{2m}}(\|g\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}).$$

Proof. It is sufficient to prove the result for $R = 1$. For $\sigma \in (0,1)$ let $\eta \in C_0^{2m}(B_1)$ be a cut-off function with $0 \le \eta \le 1$, $\eta \equiv 1$ in B_σ, $\eta \equiv 0$ for $|x| \ge \sigma'$ where $\sigma' = \frac{1+\sigma}{2}$ and $|D^\gamma \eta| \le \left(\frac{4}{1-\sigma}\right)^{|\gamma|}$ for $|\gamma| \le 2m$. Then

$$L(u\eta) = g\eta + \sum_{\substack{|\beta| \le 2m-1 \\ |\gamma| \le 2m-|\beta|}} c_{\beta,\gamma}(x) D^\beta u D^\gamma \eta \text{ in } \Omega,$$

where $c_{\beta,\gamma}$ are bounded functions with $\|c_{\beta,\gamma}\|_\infty \leq C_1$ and $C_1 = C(m)\max\{\|b_\alpha\|_\infty, \|a_{ij}\|_{C^{2m-2}}\}$. By Theorem 5

$$\begin{aligned}\|\nabla^{2m}u\|_{L^p(\Omega\cap B_\sigma)} &\leq C_2\Big(\|g\|_{L^p(\Omega)} + \sum_{\substack{0\leq k\leq 2m-1\\ 0\leq l\leq 2m-k}} \|\nabla^k u\|_{L^p(\Omega\cap B_{\sigma'})}(1-\sigma)^{-l}\Big)\\ &\leq C_3\Big(\|g\|_{L^p(\Omega)} + \sum_{k=0}^{2m-1} \|\nabla^k u\|_{L^p(\Omega\cap B_{\sigma'})}(1-\sigma)^{k-2m}\Big).\end{aligned}$$

If we introduce for $k \in \mathbb{N}_0$ the weighted norm $\Phi_k = \sup_{0<\sigma<1}(1-\sigma)^k\|\nabla^k u\|_{L^p(\Omega\cap B_\sigma)}$ then the last inequality implies

$$\Phi_{2m} \leq C_3\Big(\|g\|_{L^p(\Omega)} + \sum_{k=0}^{2m-1}\Phi_k\Big). \tag{2.3}$$

Recall the standard interpolation inequality, see Adams, Fournier [1], for $0 \leq k \leq 2m-1$

$$\|\nabla^k u\|_{L^p(\Omega\cap B_\sigma)} \leq \epsilon\|\nabla^{2m}u\|_{L^p(\Omega\cap B_\sigma)} + C\epsilon^{\frac{-k}{2m-k}}\|u\|_{L^p(\Omega\cap B_\sigma)},$$

where C is homothety invariant and hence independent of σ. Using this we find that for every fixed $\delta > 0$ there exists $\sigma(\delta) \in (0,1)$ such that

$$\begin{aligned}\Phi_k &\leq (1-\sigma)^k\|\nabla^k u\|_{L^p(\Omega\cap B_\sigma)} + \delta\\ &\leq (1-\sigma)^k\Big((1-\sigma)^{2m-k}\epsilon\|\nabla^{2m}u\|_{L^p(\Omega\cap B_\sigma)} + C\epsilon^{\frac{-k}{2m-k}}(1-\sigma)^{-k}\|u\|_{L^p(\Omega\cap B_\sigma)}\Big) + \delta\\ &= \epsilon(1-\sigma)^{2m}\|\nabla^{2m}u\|_{L^p(\Omega\cap B_\sigma)} + C\epsilon^{\frac{-k}{2m-k}}\|u\|_{L^p(\Omega\cap B_\sigma)} + \delta.\end{aligned}$$

Since $\delta > 0$ was arbitrary, we see that $\Phi_k \leq \epsilon\Phi_{2m} + C_\epsilon\Phi_0$. Hence it follows from (2.3) that $\Phi_{2m} \leq C_4(\|g\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)})$, i.e.,

$$\|\nabla^{2m}u\|_{L^p(\Omega\cap B_\sigma)} \leq \frac{C_4}{(1-\sigma)^{2m}}(\|g\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}).$$

Using the interpolation inequality again we obtain the claim. □

Proof of Theorem 1. It is convenient to rewrite the operator L in the form

$$L = (-1)^m\sum_{|\alpha|=2m} a_\alpha(x)D^\alpha + \sum_{|\alpha|\leq 2m-1} c_\alpha(x)D^\alpha.$$

Here $a_\alpha(x) = \sum_{I\in\mathcal{M}_\alpha} a_{i_1i_2}(x)\cdot a_{i_3i_4}(x)\cdots a_{i_{2m-1}i_{2m}}(x)$, where $\mathcal{M}_\alpha$ is the set of all vectors $I = (i_1,\ldots,i_{2m}) \in \{1,\ldots,N\}^{2m}$ satisfying $\#\{j\ :\ i_j = l\} = \alpha_l$ for $l = 1,\ldots,N$. Hence a_α is continuous on $\overline{\Omega}$ and $c_\alpha \in L^\infty(\Omega)$. Assume for contradiction that there exists a sequence u_k of solutions of (1.1) with $M_k := \|u_k\|_\infty \to \infty$ as $k \to \infty$. By considering a suitable subsequence we can assume that there exists $x_k \in \Omega$ such that either $M_k = u_k(x_k)$ for all $k \in \mathbb{N}$ (positive blow-up) or $M_k = -u_k(x_k)$ for all $k \in \mathbb{N}$ (negative blow-up). Define

$$v_k(y) := \frac{1}{M_k}u_k(M_k^{\frac{1-q}{2m}}y + x_k).$$

Then $\|v_k\|_\infty = 1$ and either $v_k(0) = 1$ for all $k \in \mathbb{N}$ (positive blow-up) or $v_k(0) = -1$ for all $k \in \mathbb{N}$ (negative blow-up). We may also assume that $x_k \to \bar{x} \in \overline{\Omega}$.

<u>Case 1:</u> $\bar{x} \in \Omega$. In this case v_k is well-defined on the sequence of balls $B_{\rho_k}(0)$ with $\rho_k := M_k^{\frac{q-1}{2m}} \operatorname{dist}(x_k, \partial\Omega) \to \infty$ as $k \to \infty$. Note that

$$D^\alpha v_k(y) = M_k^{\frac{1-q}{2m}|\alpha|-1} (D^\alpha u_k)(M_k^{\frac{1-q}{2m}} y + x_k).$$

For $y \in B_{\rho_k}(0)$ let

$$\bar{a}_\alpha^k(y) := a_\alpha(M_k^{\frac{1-q}{2m}} y + x_k), \quad \bar{c}_\alpha^k(y) := M_k^{(q-1)(\frac{|\alpha|}{2m}-1)} c_\alpha(M_k^{\frac{1-q}{2m}} y + x_k) \tag{2.4}$$

and define the operator

$$\bar{L}^k := (-1)^m \sum_{|\alpha|=2m} \bar{a}_\alpha^k(y) D^\alpha + \sum_{|\alpha|\le 2m-1} \bar{c}_\alpha^k(y) D^\alpha. \tag{2.5}$$

The function v_k satisfies

$$\bar{L}^k v_k(y) = f_k(y) \text{ in } B_{\rho_k}(0), \qquad \text{where } f_k(y) := \frac{1}{M_k^q} f(M_k^{\frac{1-q}{2m}} y + x_k, M_k v_k(y)). \tag{2.6}$$

By our assumption on the nonlinearity $f(x,s)$ we have that $\|f_k\|_{L^\infty(B_{\rho_k}(0))}$ is bounded in k. Note that while the ellipticity constant and the L^∞-norm of the coefficients of $\bar{L}^k$ are the same as for L, the modulus of continuity of $\bar{a}_\alpha^k$ is smaller than that of a_α. By applying Corollary 6 on the ball $B_R(0)$ for any $R > 0$ and any $p \ge 1$ there exists a constant $C_{p,R} > 0$ such that

$$\|v_k\|_{W^{2m,p}(B_R(0))} \le C_{p,R} \text{ uniformly in } k.$$

For large enough p we may extract a subsequence (again denoted v_k) such that $v_k \to v$ in $C^{2m-1,\alpha}(B_R(0))$ as $k \to \infty$ for every $R > 0$, where $v \in C^{2m-1,\alpha}_{loc}(\mathbb{R}^N)$ is bounded with $\|v\|_\infty = 1 = \pm v(0)$. Taking yet another subsequence we may assume that $f_k \overset{*}{\rightharpoonup} F$ in $L^\infty(K)$ as $k \to \infty$ for every compact set $K \subset \mathbb{R}^N$. Also we see that

$$F(y) = \begin{cases} h(\bar{x}) v(y)^q & \text{if } v(y) > 0, \\ k(\bar{x}) |v(y)|^q & \text{if } v(y) < 0, \end{cases} \tag{2.7}$$

because, e.g., if $v(y) > 0$ then there exists k_0 such that $v_k(y) > 0$ for $k \ge k_0$ and hence $M_k v_k(y) \to \infty$ as $k \to \infty$. Therefore the assumption on $f(x,s)$ implies that $f_k(y) \to h(\bar{x}) v(y)^q$ as $k \to \infty$, and a similar pointwise convergence holds at points where $v(y) < 0$. Finally, note that the pointwise convergence of f_k on the set $Z^+ = \{y \in \mathbb{R}^N : v(y) > 0\}$ and $Z^- = \{y \in \mathbb{R}^N : v(y) < 0\}$ determine due to the dominated convergence theorem the weak$*$-limit F of f_k on the set $Z^+ \cup Z^-$. Since $\bar{b}_\alpha^k(y) \to 0$ and $\bar{a}_\alpha^k(y) \to a_\alpha(\bar{x})$ as $k \to \infty$ and since we may assume that $v_k \to v$ in $W^{m,p}_{loc}(\mathbb{R}^N)$ we find that v is a bounded, weak $W^{m,p}_{loc}(\mathbb{R}^N)$-solution of

$$\mathcal{L} v = F \text{ in } \mathbb{R}^N, \qquad \text{where} \quad \mathcal{L} = (-1)^m \sum_{|\alpha|=2m} a_\alpha(\bar{x}) D^\alpha = \Big(-\sum_{i,j=1}^N a_{ij}(\bar{x}) \frac{\partial^2}{\partial y_i \partial y_j}\Big)^m. \tag{2.8}$$

Since $F \in L^\infty(\mathbb{R}^N)$ we get that $v \in W^{2m,p}_{loc}(\mathbb{R}^N) \cap C^{2m-1,\alpha}_{loc}(\mathbb{R}^N)$ is a bounded, strong solution of (2.8). Because $D^{2m}v = 0$ a.e. on the set $\{y \in \mathbb{R}^N : v(y) = 0\}$ we see that v is a strong solution of

$$\mathcal{L}v = \begin{cases} h(\bar{x})v(y)^q & \text{if } v(y) > 0, \\ 0 & \text{if } v(y) = 0, \\ k(\bar{x})|v(y)|^q & \text{if } v(y) < 0 \end{cases}$$

in $\mathbb{R}^N$. Notice that the right-hand side of the equation is $C^1(\mathbb{R}^N)$. Hence v is a classical $C^{2m,\alpha}_{loc}(\mathbb{R}^N)$ solution. By a linear change of variables we may assume that v solves

$$(-\Delta)^m v = g(v) \text{ in } \mathbb{R}^N, \qquad \text{where } g(s) = \begin{cases} h(\bar{x})s^q & \text{if } s \geq 0, \\ k(\bar{x})|s|^q & \text{if } s \leq 0. \end{cases} \tag{2.9}$$

By Lemma 15 of the Appendix we find that $v \geq 0$. This already excludes negative blow-up and implies that $g(v(y)) = h(\bar{x})v(y)^q$, $v(0) = 1$. Theorem 3 tells us that this is impossible. This finishes the contradiction argument in the first case.

<u>Case 2:</u> $\bar{x} \in \partial\Omega$. By flattening the boundary through a local change of coordinates we may assume that near $\bar{x} = 0$ the boundary is contained in the hyperplane $x_1 = 0$, and that $x_1 > 0$ corresponds to points inside Ω. Since $\partial\Omega$ is locally a C^{2m}-manifold, this change of coordinates transforms the operator L into a similar operator which satisfies the same hypotheses as L. For simplicity we call the transformed variables x and the transformed operator L. Now the function v_k is well-defined on the set $B_{\rho_k}(0) \cap \{y \in \mathbb{R}^N : y_1 > -M_k^{\frac{q-1}{2m}} x_{k,1}\}$. Since

$$1 = |\underbrace{v_k(0)}_{=\pm 1} - \underbrace{v_k(-M_k^{\frac{q-1}{2m}} x_{k,1}, 0, \ldots, 0)}_{=0}| \leq M_k^{\frac{q-1}{2m}} x_{k,1} \|\nabla v_k\|_\infty$$

we see that either $M_k^{\frac{q-1}{2m}} x_{k,1}$ is unbounded and we can conclude as in Case 1, or (by extracting a subsequence) $\tau_k := M_k^{\frac{q-1}{2m}} x_{k,1} \to \tau > 0$ as $k \to \infty$. In this case we make a further change of coordinates and define

$$\begin{aligned} w_k(z) &:= v_k(z_1 - \tau_k, z_2, \ldots, z_N), \\ \tilde{a}^k_\alpha(z) &:= \bar{a}^k_\alpha(z_1 - \tau_k, z_2, \ldots, z_N), \\ \tilde{c}^k_\alpha(z) &:= \bar{c}^k_\alpha(z_1 - \tau_k, z_2, \ldots, z_N) \end{aligned}$$

and likewise the operator $\tilde{L}^k$. Note that $w_k(\tau_k, 0, \ldots, 0) = \pm 1$. Let $\mathbb{R}^N_+ = \{z \in \mathbb{R}^N : z_1 > 0\}$ and $B^+_R = B_R(0) \cap \mathbb{R}^N_+$ for $R > 0$. For k sufficiently large the coefficients $\tilde{a}^k_\alpha$, $\tilde{c}^k_\alpha$ and the operator $\tilde{L}^k$ are well-defined in B^+_R. As before w_k satisfies

$$\tilde{L}^k w_k(z) = \tilde{f}_k(z) \text{ in } B^+_R, \qquad \text{where } \tilde{f}_k(z) := \frac{1}{M_k^q} f(M_k^{\frac{1-q}{2m}} z + (0, x_{k,2}, \ldots, x_{k,n}), M_k w_k(z)).$$

together with Dirichlet-boundary conditions on $\{z \in \mathbb{R}^N : |z| < R, z_1 = 0\}$. Hence we may apply Corollary 6 on the half-ball B^+_R for any $R > 0$ and find that for any $p \geq 1$ there exists

a constant $C_{p,R} > 0$ such that

$$\|w_k\|_{W^{2m,p}(B_R^+)} \leq C_{p,R} \text{ uniformly in } k.$$

As in Case 1 we can extract convergent subsequences $w_k \to w$ in $C^{2m-1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$ and $f_k \overset{*}{\rightharpoonup} F$ in $L^\infty(\mathbb{R}^N_+)$ as $k \to \infty$, where $F \geq 0, \not\equiv 0$ is determined in the same way as in Case 1. This time, w is a bounded, strong $W^{2m,p}_{loc}(\mathbb{R}^N_+) \cap C^{2m-1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$-solution of

$$\mathcal{L}w = F \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial}{\partial z_1} w = \ldots = \frac{\partial^{m-1}}{\partial z_1^{m-1}} w = 0 \text{ on } \partial\mathbb{R}^N_+$$

with $\mathcal{L}$ as in (2.8). By a linear change of variables we may assume that w solves

$$(-\Delta)^m w = g(w) \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial}{\partial z_1} w = \ldots = \frac{\partial^{m-1}}{\partial z_1^{m-1}} w = 0 \text{ on } \partial\mathbb{R}^N_+, \tag{2.10}$$

where g is defined as in (2.9) of Case 1. The representation formula of Theorem 9 shows that w is positive and that $g(w(z)) = h(\bar{x}) w(z)^q$. Therefore w is a positive, bounded and classical solution C^{2m}-solution of $(-\Delta)^m w = h(\bar{x}) w^q$ in $\mathbb{R}^N_+$ with Dirichlet boundary conditions and $w(0) = 1$. A contradiction is reached by Theorem 4. □

Proof of Remark 2. Take sequences of solutions (u_k, λ_k) such that $M_k := \|u_k\|_\infty \to \infty$ as $k \to \infty$, $\lambda_k \geq \lambda_0 > 0$ and define the rescaled functions

$$v_k(y) := \frac{1}{M_k} u_k\left(\left(\frac{M_k^{1-q}}{\lambda_k}\right)^{1/2m} y + x_k\right).$$

Due to the assumption $\lambda_k \geq \lambda_0 > 0$ one has that $M_k^{1-q}/\lambda_k \to 0$ as $k \to \infty$. Define further

$$\begin{aligned}
\bar{a}^k_\alpha(y) &:= a_\alpha\left(\left(\frac{M_k^{1-q}}{\lambda_k}\right)^{1/2m} y + x_k\right), \\
\bar{c}^k_\alpha(y) &:= M_k^{(q-1)(\frac{|\alpha|}{2m}-1)} \lambda_k^{\frac{|\alpha|}{2m}-1} c_\alpha\left(\left(\frac{M_k^{1-q}}{\lambda_k}\right)^{1/2m} y + x_k\right)
\end{aligned}$$

with the corresponding operator $\bar{L}^k$. Then v_k satisfies

$$\bar{L}^k v_k(y) = f_k(y) \quad \text{ where } f_k(y) := \frac{1}{M_k^q \lambda_k} f\left(\left(\frac{M_k^{1-q}}{\lambda_k}\right)^{1/2m} y + x_k, M_k v_k(y)\right).$$

Note that $\lim_{k\to\infty} f_k(y) = h(\bar{x}) v(y)^q$ on Z^+ and similarly on Z^-. The rest of the proof is as before. □

3. Green representation

The main result of this section is Theorem 9. There we state conditions on a function u on the half-space $\mathbb{R}^N_+$ under which the Green representation formula

$$u(x) = \int_{\mathbb{R}^N_+} G^+_\infty(x,y)(-\Delta)^m u(y)\, dy \text{ for all } x \in \mathbb{R}^N_+$$

holds. Here G^+_∞ is the half-space Green function, see (3.1) below. In the next section, in the proof of Theorem 4, this representation formula will be applied to solutions of (1.4).

Let us fix some notation. We recall Boggio's celebrated formula [6] for the Green function of the operator $(-\Delta)^m$ with Dirichlet boundary conditions on the unit ball $\mathbf{B} = \{x \in \mathbb{R}^N : |x| < 1\}$:

$$\begin{aligned} G_1(x,y) &= k_N^m |x-y|^{2m-N} \int_1^{(\psi(x,y)+1)^{1/2}} \frac{(z^2-1)^{m-1}}{z^{N-1}}\,dz \\ &= \frac{k_N^m}{2}|x-y|^{2m-N} \int_0^{\psi(x,y)} \frac{z^{m-1}}{(z+1)^{N/2}}\,dz \quad \text{with} \quad \psi(x,y) = \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2} \end{aligned}$$

for $x, y \in \mathbf{B}$. Here k_N^m is a suitable normalization constant. By dilation we find the Green function for the ball $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ as follows

$$\begin{aligned} G_R(x,y) &= R^{2m-N} G_1\left(\frac{x}{R}, \frac{y}{R}\right) \\ &= \frac{k_N^m}{2}|x-y|^{2m-N} \int_0^{\psi_R(x,y)} \frac{z^{m-1}}{(z+1)^{N/2}}\,dz \quad \text{with} \quad \psi_R(x,y) = \frac{(R^2-|x|^2)(R^2-|y|^2)}{R^2|x-y|^2}. \end{aligned}$$

Next we set $P_R := (R, 0, \dots, 0) \in \mathbb{R}^N_+$ and we denote by $B_R^+ := \{x \in \mathbb{R}^N : |x - P_R| < R\}$ the ball of radius R shifted by P_R. If we let G_R^+ denote the Green function on B_R^+ with respect to Dirichlet boundary conditions then we find the explicit formula

$$G_R^+(x,y) = R^{2m-N} G_1\left(\frac{x-P_R}{R}, \frac{y-P_R}{R}\right) = \frac{k_N^m}{2}|x-y|^{2m-N} \int_0^{\psi_R^+(x,y)} \frac{z^{m-1}}{(z+1)^{N/2}}\,dz$$

with

$$\psi_R^+(x,y) = \frac{(R^2-|x-P_R|^2)(R^2-|y-P_R|^2)}{R^2|x-y|^2}, \qquad x, y \in B_R.$$

Finally, if we let G^+_∞ denote the Green function of the operator $(-\Delta)^m$ on the half-space $\mathbb{R}^N_+$ subject to Dirichlet boundary conditions then

$$G^+_\infty(x,y) = \frac{k_N^m}{2}|x-y|^{2m-N} \int_0^{\psi_\infty(x,y)} \frac{z^{m-1}}{(z+1)^{N/2}}\,dz \quad \text{with} \quad \psi_\infty(x,y) = \frac{4x_1y_1}{|x-y|^2} \tag{3.1}$$

for $x, y \in \mathbb{R}^N_+$.

Lemma 7. *The Green function G_R^+ on B_R^+ converges pointwise and monotonically to the Green function G^+_∞ on $\mathbb{R}^N_+$.*

Proof. The pointwise convergence is easily checked. Let $x, y \in B_R^+$. The monotonicity of $G_R^+(x,y)$ with respect to R is equivalent to the monotonicity of $\psi_R^+(x,y)$ with respect to R. Thus

$$\begin{aligned} &\frac{d}{dR}\frac{(R^2-|x-P_R|^2)(R^2-|y-P_R|^2)}{R^2} \\ &= -\frac{2}{R^3}(R^2-|x-P_R|^2)(R^2-|y-P_R|^2) + \frac{2x_1}{R^2}(R^2-|y-P_R|^2) + \frac{2y_1}{R^2}(R^2-|x-P_R|^2). \end{aligned}$$

Setting $a := (x - P_R)/R$, $b := (y - P_R)/R$ we have $|a|^2, |b|^2 \le 1$ and we obtain from the previous computation

$$\begin{aligned}
&\frac{d}{dR}\frac{(R^2 - |x - P_R|^2)(R^2 - |y - P_R|^2)}{R^2}\\
&\qquad = 2R\Big(- (1 - |a|^2)(1 - |b|^2) + (1 + a_1)(1 - |b|^2) + (1 + b_1)(1 - |a|^2)\Big)\\
&\qquad\qquad = R\Big(\underbrace{(a_1 + \frac{1}{2} + \frac{1}{2}|a|^2)}_{\frac{1}{2}(|a|^2+2a_1+1)}(1 - |b|^2) + \underbrace{(b_1 + \frac{1}{2} + \frac{1}{2}|b|^2)}_{\frac{1}{2}(|b|^2+2b_1+1)}(1 - |a|^2)\Big),
\end{aligned}$$

and clearly $|a|^2 + 2a_1 + 1 = |(a_1 + 1, a_2, \dots, a_N)|^2 \ge 0$. This establishes the proof. $\square$

In [15], Lemma 3.4, Grunau and Sweers proved the following estimates for the polyharmonic Green function G_1 on the unit ball if $|k| \ge m$ and $x \in \mathbf{B}$, $y \in \partial\mathbf{B}$:

$$|D^k_y G_1(x, y)| \le C_{k,N,m}|x - y|^{m-N-|k|}(1 - |x|)^m \tag{3.2}$$

for some constant $C_{k,N,m} > 0$. For the Green function G_R on B_R and G^+_R on B^+_R the estimate (3.2) transforms as follows:

$$|D^k_y G_R(x, y)| \le C_{k,N,m}|x - y|^{m-N-|k|}(R - |x|)^m \tag{3.3}$$

if $x \in B_R, y \in \partial B_R$. Likewise,

$$|D^k_y G^+_R(x, y)| \le C_{k,N,m}|x - y|^{m-N-|k|}|x|^m \tag{3.4}$$

if $x = (x_1, 0, \dots, 0) \in B^+_R$ with $x_1 \in (0, R), y \in \partial B^+_R$.

Lemma 8. *Let G be the Green function of $(-\Delta)^m$ with Dirichlet boundary condition on an arbitrary ball $B \subset \mathbb{R}^n$ with exterior unit normal ν on ∂B. For any function $v \in C^{2m-1}(\overline{B}) \cap W^{2m,p}(B)$ with $p > \frac{N}{2m}$ one has the following Poisson-Green representation for $x \in B$: for m even*

$$\begin{aligned}
v(x) = &\sum_{i=1}^{m/2} \oint_{\partial B} \Big(\Delta^{i-1} v(y)\partial_{\nu_y}\Delta^{m-i}_y G(x, y) - \Delta^{m-i}_y G(x, y)\partial_{\nu_y}\Delta^{i-1} v(y)\Big)\, ds_y\\
&+ \int_B G(x, y)(-\Delta)^m v(y)\, dy.
\end{aligned} \tag{3.5}$$

and for m odd

$$\begin{aligned}
v(x) = -&\sum_{i=1}^{(m-1)/2} \oint_{\partial B} \Big(\Delta^{i-1} v(y)\partial_{\nu_y}\Delta^{m-i}_y G(x, y) - \Delta^{m-i}_y G(x, y)\partial_{\nu_y}\Delta^{i-1} v(y)\Big)\, ds_y\\
&- \oint_{\partial B} \Delta^{(m-1)/2} v(y)\partial_{\nu_y}\Delta^{(m-1)/2}_y G(x, y)\, ds_y + \int_B G(x, y)(-\Delta)^m v(y)\, dy.
\end{aligned} \tag{3.6}$$

Proof. First assume $v \in C^{2m}(\overline{B})$. Consider the identity

$$\begin{aligned}\sum_{i=1}^m \operatorname{div}\left(\Delta^{i-1}v\nabla\Delta^{m-i}G - \Delta^{m-i}G\nabla\Delta^{i-1}v\right) &= \sum_{i=1}^m \left(\Delta^{i-1}v\Delta^{m-i+1}G - \Delta^{m-i}G\Delta^i v\right)\\ &= v\Delta^m G - G\Delta^m v \text{ in } B.\end{aligned}$$

If we integrate this identity over B and take into account that $D_y^\alpha G(x,y) = 0$ for $|\alpha| \leq m-1$ and $x \in B, y \in \partial B$ then we obtain the claim. For $v \in C^{2m-1}(\overline{B}) \cap W^{2m,p}(B)$ we can argue by approximation and Lebesgue's dominated convergence theorem if we take into account that $\int_B G(x,y)|h(y)|\,dy \leq \text{const.}\,\|h\|_{L^p(B)}$ provided $h \in L^p(B)$ and $p > \frac{N}{2m}$. $\square$

Theorem 9. *Suppose that* $u \in C^{2m-1}(\overline{\mathbb{R}^N_+}) \cap W^{2m,p}_{loc}(\mathbb{R}^N_+)$, $p > \frac{N}{2m}$ *is a function with the following properties:*

(i) *u and all partial derivatives of u of order less than or equal to $2m-1$ are bounded,*
(ii) *u satisfies Dirichlet boundary conditions on $\partial\mathbb{R}^N_+$,*
(iii) *$(-\Delta)^m u \in L^p_{loc}(\mathbb{R}^N_+)$ is non-negative in $\mathbb{R}^N_+$.*

Then

$$u(x) = \int_{\mathbb{R}^N_+} G^+_\infty(x,y)(-\Delta)^m u(y)\,dy \qquad \text{for every } x \in \mathbb{R}^N_+. \tag{3.7}$$

Proof. Let us first consider the case where m is even. It clearly suffices to prove (3.7) for $x = (x_1, 0, \dots, 0) \in \mathbb{R}^N_+$ with $x_1 > 0$ fixed. In the following we consider $R > 2x_1$. Then $x \in B^+_R$, $x_1 \in (0,R)$, and (3.3) yields for $y \in \partial B^+_R$ and $i \leq \frac{m}{2}$ the following estimates:

$$\begin{aligned}|\Delta_y^{m-i}G^+_R(x,y)| &\leq C_{i,N,m}|x-y|^{-m-N+2i}|x|^m,\\ |\partial_{\nu_y}\Delta_y^{m-i}G^+_R(x,y)| &\leq C_{i,N,m}|x-y|^{-m-N+2i-1}|x|^m.\end{aligned}$$

Combining this with (3.5), we get

$$\begin{aligned}&\left|\int_{B^+_R} G^+_R(x,y)(-\Delta)^m u(y)\,dy - u(x)\right|\\ &\quad\leq C|x|^m \sum_{i=1}^{m/2} \oint_{\partial B^+_R} \Big(|\Delta^{i-1}u(y)||x-y|^{-m-N+2i-1} + |x-y|^{-m-N+2i}|\partial_{\nu_y}\Delta^{i-1}u(y)|\Big)\,ds_y.\end{aligned}$$

Since $|x-y| \geq |x|$ for $y \in \partial B^+_R$, we conclude that

$$\begin{aligned}\left|\int_{B^+_R} G^+_R(x,y)(-\Delta)^m u(y)\,dy - u(x)\right| &\leq C_x \sum_{i=1}^{m/2} \oint_{\partial B^+_R} |x-y|^{-N}\left(|\Delta^{i-1}u(y)| + |\partial_{\nu_y}\Delta^{i-1}u(y)|\right) ds_y\\ &= C_x \sum_{i=0}^{m/2-1} \oint_{\partial B^+_R} |x-y|^{-N}\Big(|\Delta^i u(y)| + |\partial_{\nu_y}\Delta^i u(y)|\Big)\,ds_y.\end{aligned} \tag{3.8}$$

We claim that, for every $x \in \mathbb{R}^N_+$,

$$\sum_{i=0}^{m/2-1} \oint_{\partial B_R^+} |x-y|^{-N}\Big(|\Delta^i u(y)| + |\partial_{\nu_y}\Delta^i u(y)|\Big)\, ds_y \to 0 \qquad \text{as } R\to\infty. \tag{3.9}$$

For $N=1$ this is obvious since

$$\oint_{\partial B_R^+} |x-y|^{-N}\Big(|\Delta^i u(y)| + |\partial_{\nu_y}\Delta^i u(y)|\Big)\, ds_y = |x-2R|^{-1}\Big(|u^{(2i)}(2R)| + |u^{(2i+1)}(2R)|\Big)$$

for $i \le \frac{m}{2}-1$ as a consequence of the boundary conditions. For $N\ge 2$ and $0\le a\le b\le 2R$ let us consider the set $B(a,b) := \{y\in\partial B_R^+ \,:\, a\le y_1\le b\}$. On $B(a,b)$ we have $y=(y_1,y')$ with $|y'| = \sqrt{2Ry_1-y_1^2}$. For $N\ge 3$ we parameterize $B(a,b)$ by the map

$$F:\begin{cases} (a,b)\times\mathbb{S}^{N-2} & \to \quad B(a,b),\\ (y_1,\varphi) & \mapsto \quad \Big(y_1, \sqrt{2Ry_1-y_1^2}\,\theta(\varphi)\Big),\end{cases}$$

where $\varphi = (\varphi_2,\dots,\varphi_{N-1})$ and

$$\theta(\varphi) = \begin{pmatrix} \cos\varphi_2\sin\varphi_3\sin\varphi_4\dots\sin\varphi_{N-1}\\ \sin\varphi_2\sin\varphi_3\sin\varphi_4\dots\sin\varphi_{N-1}\\ \cos\varphi_3\sin\varphi_4\dots\sin\varphi_{N-1}\\ \vdots\\ \cos\varphi_{N-2}\sin\varphi_{N-1}\\ \cos\varphi_{N-1}\end{pmatrix}.$$

Let $DF = (b_1|b_2|\cdots|b_{N-1})$ be the Jacobian Matrix of the map F and $\mathrm{gr}(DF) = \det(DF^T\cdot DF) = \det(b_i\cdot b_j)_{i,j=1,\dots,N-1}$ be the Gram determinant of DF. Since $b_1 = \big(1, \frac{R-y_1}{\sqrt{2Ry_1-y_1^2}}\theta\big)^T$ and $b_i = \big(0, \sqrt{2Ry_1-y_1^2}\frac{\partial\theta}{\partial\varphi_i}\big)^T$ for $i=2,\dots,N-1$ we find $b_1\cdot b_1 = \frac{R^2}{2Ry_1-y_1^2}$, $b_1\cdot b_i = 0$ for $i=2,\dots,N-1$. Therefore

$$\sqrt{\mathrm{gr}(DF)} = \sqrt{b_1\cdot b_1}\cdot(2Ry_1-y_1^2)^{\frac{N-2}{2}}|\det(D\theta)| = R(2Ry_1-y_1^2)^{\frac{N-3}{2}}\cdot|\det(D\theta)|.$$

Since

$$\det(D\theta) = (-1)^{N-1}\sin\varphi_3(\sin\varphi_4)^2\dots(\sin\varphi_{N-1})^{N-3}$$

we obtain finally

$$\sqrt{\mathrm{gr}(DF)} \le R(2Ry_1-y_1^2)^{\frac{N-3}{2}}.$$

Therefore we can write the surface integral as follows (if $N=2$ the line integral is parameterized by $(y_1, \pm\sqrt{2Ry_1-y_1^2})^T$)

$$\oint_{B(a,b)} |x-y|^{-N} ds_y = \begin{cases} \displaystyle\int_a^b\int_{\mathbb{S}^{N-2}} |x-F(y_1,\varphi)|^{-N}\sqrt{\mathrm{gr}\,DF}\,d\varphi\,dy_1 & \text{if } N\ge 3,\\ \displaystyle 2\int_a^b \frac{R(2Ry_1-y_1^2)^{-1/2}}{x_1^2+2Ry_1-2x_1y_1}\,dy_1 & \text{if } N=2.\end{cases}$$

Since $|x-F(y_1,\varphi)|^2 = x_1^2+2Ry_1-2x_1y_1$ and $b\le 2R$, we can now estimate as follows:

$$
\begin{aligned}
\oint_{B(a,b)} |x-y|^{-N} ds_y &\le c_1 \int_a^{2R} \frac{R(2Ry_1-y_1^2)^{\frac{N-3}{2}}}{(x_1^2+2Ry_1-2x_1y_1)^{\frac{N}{2}}}\,dy_1 = \frac{c_1}{R^2}\int_a^{2R} \frac{(2\frac{y_1}{R}-(\frac{y_1}{R})^2)^{\frac{N-3}{2}}}{(\frac{x_1^2}{R^2}+2\frac{y_1}{R}(1-\frac{x_1}{R}))^{\frac{N}{2}}}\,dy_1\\
&= \frac{c_1}{R}\int_{\frac{a}{R}}^2 \frac{(2t-t^2)^{\frac{N-3}{2}}}{(\frac{x_1^2}{R^2}+2t(1-\frac{x_1}{R}))^{\frac{N}{2}}}\,dt \le \frac{c_1}{R}\int_{\frac{a}{R}}^2 \frac{t^{\frac{N-3}{2}}(2-t)^{\frac{N-3}{2}}}{(\frac{x_1^2}{R^2}+t)^{\frac{N}{2}}}\,dt\\
&= \frac{c_1}{R}\int_{\frac{a}{R}}^2 \frac{t^{-\frac{1}{2}}(2-t)^{\frac{N-3}{2}}}{\frac{x_1^2}{R^2}+t}\left(\frac{t}{\frac{x_1^2}{R^2}+t}\right)^{\frac{N-2}{2}} dt \le \frac{c_2}{R}\int_{\frac{a}{R}}^2 \frac{t^{-\frac{1}{2}}(2-t)^{-\frac{1}{2}}}{\frac{x_1^2}{R^2}+t}\,dt
\end{aligned}
$$

with $c_2 = 2^{\frac{N-2}{2}}c_1$. Here we have also used that $R\ge 2x_1$ and $N\ge 2$. From now on we assume $a\le R$ and split the remaining integral as follows:

$$
\oint_{B(a,b)} |x-y|^{-N} ds_y \le \underbrace{\frac{c_2}{R}\int_{\frac{a}{R}}^1 \frac{t^{-1/2}(2-t)^{-1/2}}{\frac{x_1^2}{R^2}+t}\,dt}_{=:I_1} + \underbrace{\frac{c_2}{R}\int_1^2 \frac{t^{-1/2}(2-t)^{-1/2}}{\frac{x_1^2}{R^2}+t}\,dt}_{=:I_2}. \tag{3.10}
$$

In the first integral I_1 we have $(2-t)\ge 1$ and therefore

$$
I_1 \le \frac{c_2}{R}\int_{\frac{a}{R}}^1 \frac{t^{-1/2}}{\frac{x_1^2}{R^2}+t}\,dt.
$$

If $a>0$, we conclude that

$$
I_1 \le \frac{c_2}{R}\int_{\frac{a}{R}}^1 t^{-\frac{3}{2}}\,dt \le \frac{2c_2}{\sqrt{aR}}, \tag{3.11}
$$

while for $a=0$ the substitution $z=\frac{R^2}{x_1^2}t$ yields

$$
I_1 \le \frac{c_2}{R}\int_0^1 \frac{t^{-1/2}}{\frac{x_1^2}{R^2}+t}\,dt = \frac{c_2}{x_1}\int_0^{\frac{R^2}{x_1^2}} \frac{1}{\sqrt{z}(1+z)}\,dz \le \frac{c_2}{x_1}\int_0^\infty \frac{1}{\sqrt{z}(1+z)}\,dz = \frac{c_3}{x_1}. \tag{3.12}
$$

For I_2 we have

$$
I_2 \le \frac{c_2}{R}\int_1^2 (2-t)^{-1/2}\,dt = \frac{c_4}{R}. \tag{3.13}
$$

Collecting the inequalities (3.10), (3.11), (3.12) and recalling that $R\ge 2x_1$, we obtain

$$
\oint_{B(a,b)} |x-y|^{-N} ds_y \le c_5 \begin{cases} 1/x_1 & \text{for } a=0,\\ 1/\sqrt{aR} & \text{for } a>0. \end{cases} \tag{3.14}
$$

Now let $\varepsilon>0$. By the standard mean-value theorem using the Dirichlet boundary conditions and the boundedness of the derivatives of orders up to m, there exists $\delta>0$ such that

$$
\sum_{i=0}^{m/2-1} \Big(|\Delta^i u(y)| + |\partial_{\nu_y}\Delta^i u(y)|\Big) \le \varepsilon \qquad \text{on } \{y\in\mathbb{R}^N_+ \,:\, |y_1|\le\delta\}.
$$

Hence we apply (3.14) and obtain

$$\begin{aligned}\sum_{i=0}^{m/2-1} &\oint_{\partial B_R^+} |x-y|^{-N}\Big(|\Delta^i u(y)| + |\partial_{\nu_y}\Delta^i u(y)|\Big)\, ds_y \\ &\le \varepsilon \oint_{B(0,\delta)} |x-y|^{-N}\, ds_y + c_6 \oint_{B(\delta,2R)} |x-y|^{-N}\, ds_y \\ &\le \varepsilon\frac{c_5}{x_1} + \frac{c_5 c_6}{\sqrt{\delta R}} \;\to\; \varepsilon\frac{c_5}{x_1} \qquad \text{as } R\to\infty.\end{aligned}$$

Since ε was chosen arbitrarily, we conclude that (3.9) holds.

Using (3.8), (3.9) and Lemma 7 together with the monotone convergence theorem we get

$$u(x) = \lim_{R\to\infty}\int_{B_R^+} G_R^+(x,y)(-\Delta)^m u(y)\, dy = \int_{\mathbb{R}^N_+} G_\infty^+(x,y)(-\Delta)^m u(y)\, dy.$$

This finishes the proof in the case where m is an even integer. In the case where m is odd only minor modifications are needed. We use (3.6) instead of (3.5) together with the estimates arising from (3.4): for $x\in B_R^+$, $y\in\partial B_R^+$ and $i\le\frac{m-1}{2}$,

$$|\Delta_y^{m-i} G_R^+(x,y)| \le C_{i,N,m}|x-y|^{-m-N+2i}|x|^m$$

and for $i\le\frac{m+1}{2}$,

$$|\partial_{\nu_y}\Delta_y^{m-i} G_R^+(x,y)| \le C_{i,N,m}|x-y|^{-m-N+2i-1}|x|^m.$$

By essentially the same estimates as before, we now obtain

$$\begin{aligned}&\left|\int_{B_R^+} G_R^+(x,y)(-\Delta)^m u(y)\, dy - u(x)\right| \\ &\qquad\le C_x \oint_{\partial B_R^+} |x-y|^{-N}\Big(\sum_{i=0}^{(m-1)/2}|\Delta^i u(y)| + \sum_{i=0}^{(m-3)/2}|\partial_{\nu_y}\Delta^i u(y)|\Big)\, ds_y.\end{aligned}$$

Again, as a consequence of the boundary conditions, for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\sum_{i=0}^{(m-1)/2}|\Delta^i u(y)| + \sum_{i=0}^{(m-3)/2}|\partial_{\nu_y}\Delta^i u(y)| \le \varepsilon \qquad \text{on } \{y\in\mathbb{R}^N_+ \,:\, |y_1|\le\delta\}.$$

We therefore may conclude as in the case where m is even that

$$\oint_{\partial B_R^+}|x-y|^{-N}\Big(\sum_{i=0}^{(m-1)/2}|\Delta^i u(y)| + \sum_{i=0}^{(m-3)/2}|\partial_{\nu_y}\Delta^i u(y))|\Big)\, ds_y \to 0 \qquad \text{as } R\to\infty.$$

Using again the monotone convergence theorem, we conclude that

$$u(x) = \lim_{R\to\infty}\int_{B_R^+} G_R^+(x,y)(-\Delta)^m u(y)\, dy = \int_{\mathbb{R}^N_+} G_\infty^+(x,y)(-\Delta)^m u(y)\, dy.$$

□

4. Proof of the Liouville Theorem in the half-space

This section is devoted to the proof of Theorem 4. Let $m \in \mathbb{N}$ and assume that $q > 1$ if $N \le 2m$ and $1 < q < \frac{N+2m}{N-2m}$ if $N > 2m$. Let u be a classical non-negative bounded solution of

$$(-\Delta)^m u = u^q \text{ in } \mathbb{R}^N_+, \quad u = \frac{\partial u}{\partial x_1} = \ldots = \frac{\partial^{m-1} u}{\partial x_1^{m-1}} = 0 \text{ on } \partial\mathbb{R}^N_+.$$

We need to show that $u \equiv 0$. From Theorem 9 we know that

$$u(x) = \int_{\mathbb{R}^N_+} G^+_\infty(x,y) u^q(y)\, dy \qquad \text{for every } x \in \mathbb{R}^N_+. \tag{4.1}$$

We consider the conformal diffeomorphism

$$\varphi : \mathbf{B} \to \mathbb{R}^N_+, \qquad \varphi(y) = 2\frac{y+e_1}{|y+e_1|^2} - e_1,$$

where $e_1 = (1,0,\ldots,0)$ is the first coordinate vector. The following formula shows how G^+_∞ is related to the Green function G_1 on the unit ball.

Lemma 10. $G^+_\infty(\varphi(x),\varphi(y)) = \left(\frac{2}{|x+e_1||y+e_1|}\right)^{2m-N} G_1(x,y)$ *for all* $x, y \in \mathbf{B}$.

Proof. An easy calculation yields

$$|\varphi(x) - \varphi(y)| = \frac{2|x-y|}{|x+e_1||y+e_1|} \qquad \text{for } x, y \in \mathbf{B}. \tag{4.2}$$

Considering the functions $\psi(x,y) = \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}$ and $\psi_\infty(x,y) = \frac{4x_1y_1}{|x-y|^2}$ as in Section 3, we obtain

$$\begin{aligned}\psi_\infty(\varphi(x),\varphi(y)) &= \frac{|x+e_1|^2\varphi_1(x)|y+e_1|^2\varphi_1(y)}{|x-y|^2} = \frac{(2x_1+2-|x+e_1|^2)(2y_1+2-|y+e_1|^2)}{|x-y|^2}\\ &= \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2} = \psi(x,y) \quad \text{for } x, y \in \mathbf{B}.\end{aligned}$$

We conclude that

$$\begin{aligned}G^+_\infty(\varphi(x),\varphi(y)) &= \frac{k_N^m}{2}|\varphi(x)-\varphi(y)|^{2m-N} \int_0^{\psi_\infty(\varphi(x),\varphi(y))} \frac{z^{m-1}}{(z+1)^{N/2}}\, dz\\ &= \left(\frac{2}{|x+e_1||y+e_1|}\right)^{2m-N} \frac{k_N^m}{2}|x-y|^{2m-N} \int_0^{\psi(x,y)} \frac{z^{m-1}}{(z+1)^{N/2}}\, dz\\ &= \left(\frac{2}{|x+e_1||y+e_1|}\right)^{2m-N} G_1(x,y) \qquad \text{for all } x, y \in \mathbf{B}.\end{aligned}$$

□

Corollary 11. *Define the function* $v : \mathbf{B} \to \mathbb{R}$ *by*

$$v(x) := |x+e_1|^{2m-N} u(\varphi(x))$$

and the function $h : \mathbf{B} \times [0, \infty) \to [0, \infty)$ *by*

$$h(x, t) := 2^{2m} |x + e_1|^{-\alpha}\, t^q$$

where $\alpha := N + 2m - q(N - 2m) \geq 0$ *by assumption on* q*. Then* v *satisfies*

$$v(x) = \int_{\mathbf{B}} G_1(x, y) h(y, v(y))\, dy \qquad \textit{for all } x \in \mathbf{B}. \tag{4.3}$$

Proof. The Jacobian determinant of φ satisfies $|J_\varphi(y)| = \frac{2^N}{|y+e_1|^{2N}}$ for $y \in \mathbb{R}^N_+$. Therefore we have

$$\begin{aligned} u(\varphi(x)) &= \int_{\mathbb{R}^N_+} G^+_\infty(\varphi(x), y) u^q(y)\, dy = \int_{\mathbf{B}} G^+_\infty(\varphi(x), \varphi(y)) u^q(\varphi(y)) |J_\varphi(y)|\, dy \\ &= \int_{\mathbf{B}} \left(\frac{2}{|x + e_1||y + e_1|} \right)^{2m-N} G_1(x, y) \Big(|y + e_1|^{N-2m} v(y) \Big)^q \frac{2^N}{|y + e_1|^{2N}}\, dy \\ &= \frac{1}{|x + e_1|^{2m-N}} \int_{\mathbf{B}} G_1(x, y) h(y, v(y))\, dy \end{aligned}$$

for all $x \in \mathbf{B}$, so that

$$v(x) = |x + e_1|^{2m-N} u(\varphi(x)) = \int_{\mathbf{B}} G_1(x, y) h(y, v(y))\, dy.$$

$\square$

Proposition 12. *The function* $v : \mathbf{B} \to \mathbb{R}$ *is axially symmetric with respect to the* x_1*-axis.*

Proof. We assume that $N \geq 2$ and $v \not\equiv 0$, since otherwise the statement is trivial. The integral representation (4.3) implies that v is strictly positive in $\mathbf{B}$. Note that for every $x_0 \in \partial\mathbf{B} \setminus \{-e_1\}$ we have that

$$\lim_{x \to x_0,\, x \in \mathbf{B}} v(x) = \lim_{x \to x_0,\, x \in \mathbf{B}} |x + e_1|^{2m-N} u(\varphi(x)) = |x_0 + e_1|^{2m-N} u\Big(2 \frac{x_0 + e_1}{|x_0 + e_1|^2} - e_1 \Big) = 0$$

since $2\frac{x_0+e_1}{|x_0+e_1|^2} - e_1 \in \partial\mathbb{R}^N_+$. Hence the function v – extended trivially on $\partial\mathbf{B} \setminus \{-e_1\}$ – is continuous in $\overline{B} \setminus \{-e_1\}$. We fix a unit vector $e \in \mathbb{R}^N$ perpendicular to e_1 (i.e., $|e| = 1$ and $e \cdot e_1 = 0$), and we show that v is symmetric with respect to the hyperplane $T := \{x \in \mathbb{R}^N : x \cdot e = 0\}$. For this we apply a moving plane argument based on the integral representation (4.3) and reflection inequalities derived in [4, 11] for G_1. We need some notation. For $\lambda \geq 0$, we consider the open half-space $H_\lambda = \{x \in \mathbb{R}^N : x \cdot e > \lambda\}$ and the reflection $x \mapsto x^\lambda := x - 2(x \cdot e - \lambda)e$ at the hyperplane ∂H_λ. We also consider the set

$$J_\lambda := \{x \in \mathbf{B} : x \cdot e < \lambda \text{ and } x^\lambda \notin \mathbf{B}\}$$

which has nonempty interior if $\lambda > 0$. With these definitions, the inequalities stated in [4, Lemma 4] (see also [11, Lemma 3] for the biharmonic case) translate into the following reflection inequalities:

$$\left. \begin{aligned} &G_1(x^\lambda, y^\lambda) > G_1(x, y^\lambda) \qquad \text{and} \\ &G_1(x^\lambda, y^\lambda) - G_1(x, y) > G_1(x, y^\lambda) - G_1(x^\lambda, y) \end{aligned} \right\} \qquad \text{for all } x, y \in H_\lambda \cap \mathbf{B} \tag{4.4}$$

and

$$G_1(x^\lambda, y) - G_1(x,y) > 0 \qquad \text{for } x \in H_\lambda \cap \mathbf{B},\ y \in J_\lambda. \tag{4.5}$$

Now (4.5) and the strict positivity of v in $\mathbf{B}$ imply that

$$\int_{J_\lambda} [G_1(x^\lambda, y) - G_1(x,y)]\, h(y, v(y))\, dy > 0 \qquad \text{for } \lambda > 0 \text{ and } x \in H_\lambda \cap \mathbf{B}. \tag{4.6}$$

We claim that the following reflection inequality holds for every $\lambda > 0$:

$$v(x) \le v(x^\lambda) \qquad \text{for all } x \in H_\lambda \cap \mathbf{B}. \tag{$\mathcal{C}_\lambda$}$$

We put

$$\lambda_* := \inf\{\lambda > 0 \,:\, (\mathcal{C}_{\lambda'}) \text{ holds for all } \lambda' \ge \lambda\}.$$

Then $\lambda_* \le 1$. Using the continuity of v in $\mathbf{B}$, it is easy to see that $(\mathcal{C}_{\lambda_*})$ holds. We suppose for contradiction that $\lambda_* > 0$. Since $0 < |x + e_1|^{-\alpha} \le |x^{\lambda_*} + e_1|^{-\alpha}$ for $x \in H_{\lambda_*} \cap \mathbf{B}$ and v is positive in $\mathbf{B}$, $(\mathcal{C}_{\lambda_*})$ yields

$$h(x, v(x)) \le h(x^{\lambda_*}, v(x^{\lambda_*})) \quad \text{for } x \in H_{\lambda_*} \cap \mathbf{B}. \tag{4.7}$$

We claim that

$$v(x) < v(x^{\lambda_*}) \qquad \text{for all } x \in H_{\lambda_*} \cap \mathbf{B}. \tag{4.8}$$

Indeed, by using (4.4), (4.6) and (4.7) we have

$$\begin{aligned}
v(x^{\lambda_*}) - v(x) &= \int_{\mathbf{B}} [G_1(x^{\lambda_*}, y) - G_1(x,y)]\, h(y, v(y))\, dy = \int_{H_{\lambda_*} \cap \mathbf{B}} \dots\, dy + \int_{\mathbf{B} \setminus H_{\lambda_*}} \dots\, dy \\
&= \int_{H_{\lambda_*} \cap \mathbf{B}} \Big([G_1(x^{\lambda_*}, y) - G_1(x,y)]\, h(y, v(y)) + [G_1(x^{\lambda_*}, y^{\lambda_*}) - G_1(x, y^{\lambda_*})]\, h(y^{\lambda_*}, v(y^{\lambda_*})) \Big)\, dy \\
&\quad + \int_{J_{\lambda_*}} [G_1(x^{\lambda_*}, y) - G_1(x,y)]\, h(y, v(y))\, dy \\
&> \int_{H_{\lambda_*} \cap \mathbf{B}} [G_1(x^{\lambda_*}, y^{\lambda_*}) - G_1(x, y^{\lambda_*})]\, [h(y^{\lambda_*}, v(y^{\lambda_*})) - h(y, v(y))]\, dy \ge 0 \quad \text{for } x \in H_{\lambda_*} \cap \mathbf{B}.
\end{aligned}$$

Hence (4.8) is true. For $0 < \mu \le \lambda_*$ we now consider the difference function

$$w_\mu : H_\mu \cap \mathbf{B} \to \mathbb{R}, \qquad w_\mu(x) = v(x^\mu) - v(x)$$

and the set

$$W_\mu := \{x \in H_\mu \cap \mathbf{B} \,:\, w_\mu(x) < 0\}.$$

We note that $W_{\lambda_*} = \varnothing$, and we claim that

$$|W_\mu| \to 0 \qquad \text{as } \mu \to \lambda_*, \tag{4.9}$$

where $|\cdot|$ denotes the Lebesgue measure. Indeed, let $\varepsilon \in (0, \lambda_*)$, and consider the compact set

$$K := \{x \in \mathbf{B} \,:\, x \cdot e \ge \lambda_* + \varepsilon,\ |x| \le 1 - \varepsilon\} \subset H_{\lambda_*} \cap \mathbf{B}$$

Then $\inf_{x\in K} w_{\lambda_*}(x) > 0$ by (4.8). The continuity of v in $\mathbf{B}$ implies that there exists $\lambda_1 \in (\lambda_* - \varepsilon, \lambda_*)$ such that

$$\inf_{x\in K} w_\mu(x) > 0 \quad \text{for } \lambda_1 \le \mu \le \lambda_*.$$

Hence $W_\mu \subset (H_\mu \cap \mathbf{B}) \setminus K \subset \{x \in \mathbf{B} \,:\, |x_1 - \lambda_*| \le \varepsilon \text{ or } |x| \ge 1-\varepsilon\}$ and therefore

$$|W_\mu| \le 2^N \varepsilon + \varepsilon\omega_{N-1} \qquad \text{for } \lambda_1 \le \mu \le \lambda_*,$$

where ω_{N-1} denotes the area of the $N-1$-dimensional unit sphere. Since ε was chosen arbitrarily small, (4.9) follows. Next we note that, for $\mu \ge \frac{\lambda_*}{2}$ and $x \in W_\mu$,

$$h(x^\mu, v(x^\mu)) - h(x, v(x)) = |x^\mu + e_1|^{-\alpha} v^q(x^\mu) - |x+e_1|^{-\alpha} v^q(x) \ge |x+e_1|^{-\alpha}(v^q(x^\mu) - v^q(x))$$

$$\ge \left(\frac{\lambda_*}{2}\right)^{-\alpha}[v^q(x^\mu) - v^q(x)] \ge \left(\frac{\lambda_*}{2}\right)^{-\alpha} q v^{q-1}(x)[v(x^\mu) - v(x)] \ge c(\lambda_*) w_\mu(x) \tag{4.10}$$

with $c(\lambda_*) = \left(\frac{\lambda_*}{2}\right)^{-\alpha} q\Big(\sup_{x \in H_{\lambda_*/2}\cap\mathbf{B}} v(x)\Big)^{q-1} < \infty$. We also note that

$$h(x^\mu, v(x^\mu)) - h(x, v(x)) \ge 0 \qquad \text{for } x \in H_\mu \setminus W_\mu. \tag{4.11}$$

From now on we assume that $\frac{\lambda_*}{2} \le \mu \le \lambda_*$. For $x \in W_\mu$ we use (4.4), (4.6), (4.10) and (4.11) to estimate

$$\begin{aligned}
0 > w_\mu(x) &= \int_{\mathbf{B}} [G_1(x^\mu, y) - G_1(x,y)]\, h(y, v(y))\, dy \\
&> \int_{H_\mu \cap \mathbf{B}} \Big([G_1(x^\mu, y) - G_1(x,y)]\, h(y, v(y)) + [G_1(x^\mu, y^\mu) - G_1(x, y^\mu)]\, h(y^\mu, v(y^\mu)) \Big) dy \\
&\ge \int_{H_\mu \cap \mathbf{B}} [G_1(x^\mu, y^\mu) - G_1(x, y^\mu)]\, [h(y^\mu, v(y^\mu)) - h(y, v(y))]\, dy \\
&\ge \int_{W_\mu} [G_1(x^\mu, y^\mu) - G_1(x, y^\mu)]\, [h(y^\mu, v(y^\mu)) - h(y, v(y))]\, dy \\
&\ge c(\lambda_*) \int_{W_\mu} [G_1(x^\mu, y^\mu) - G_1(x, y^\mu)] w_\mu(y)\, dy \\
&\ge c_{N,m}\, c(\lambda_*) \int_{W_\mu} |x-y|^{1-N} w_\mu(y)\, dy,
\end{aligned} \tag{4.12}$$

where in the last step we use the estimate

$$0 < G_1(x,y) \le c_{N,m}|x-y|^{1-N} \qquad \text{for } x, y \in \mathbf{B},\ x \ne y$$

with some $c_{N,m} > 0$, which is easily deduced from the integral representation of G_1 in Section 3. Next we pick $s > 1$ large enough such that $1 < q := \frac{1}{\frac{1}{N} + \frac{1}{s}} < s$. Then the Hardy-Littlewood-Sobolev inequality (see e.g. [17, Section 4.3]) implies that

$$\left(\int_{\mathbb{R}^N} \Big| \int_{W_\mu} |x-y|^{1-N} w_\mu(y)\, dy \Big|^s dx\right)^{\frac{1}{s}} = \Big\| |\cdot|^{1-N} * (1_{W_\mu} w_\mu) \Big\|_{L^s(\mathbb{R}^N)} \le c_{s,q} \|w_\mu\|_{L^q(W_\mu)}$$

with a constant $c_{s,q} > 0$. Combining this inequality with (4.12) and Hölder's inequality, we obtain

$$\|w_\mu\|_{L^s(W_\mu)} \le c_0 \|w_\mu\|_{L^q(W_\mu)} \le c_0 |W_\mu|^{\frac{s-q}{sq}} \|w_\mu\|_{L^s(W_\mu)} \qquad \text{with } c_0 := c_{N,m}\, c(\lambda_*)\, c_{s,q}. \tag{4.13}$$

Now (4.9) and (4.13) imply that $\|w_\mu\|_{L^s(W_\mu)} = 0$ if $\mu < \lambda_*$ is close enough to λ_*. Hence property $(\mathcal{C}_\mu)$ holds if $\mu < \lambda_*$ is close enough to λ_*, which contradicts the definition of λ_*. It follows that $\lambda_* = 0$, thus $(\mathcal{C}_\lambda)$ holds for all $\lambda > 0$, as claimed. By continuity, we now deduce that

$$v(x) \le v(x^0) \qquad \text{for } x \in H_0 \cap \mathbf{B}.$$

Repeating the moving plane procedure for $-e$ in place of e, we get

$$v(x) \ge v(x^0) \qquad \text{for } x \in H_0 \cap \mathbf{B}.$$

Hence equality holds, i.e., v is symmetric with respect to the hyperplane $T = \{x \in \mathbb{R}^N : x \cdot e = 0\}$ as claimed. Since e was chosen arbitrarily with $|e| = 1$ and $e \cdot e_1 = 0$ we conclude that v is axially symmetric with respect to the x_1-axis. □

Corollary 13. *The function u only depends on the x_1-variable.*

Proof. Since v is axially symmetric with respect to the x_1-axis, the same is true for u. Let $z \in \mathbb{R}^{N-1}$ be arbitrary, and consider the function

$$U_z : \mathbb{R}^N_+ \to \mathbb{R}, \qquad U_z(x_1, x') = u(x_1, x' - z) \quad \text{for } (x_1, x') \in [0, \infty) \times \mathbb{R}^{N-1}.$$

Then U_z satisfies the same assumptions as u, so it is also axially symmetric with respect to the x_1-axis. This readily implies that u only depends on the x_1-variable. □

Theorem 14. *Let $f : [0, \infty) \to [0, \infty)$ be a continuous function with $f(0) = 0$ and $f(s) > 0$ for $s > 0$. If u is a classical non-negative bounded solution of the one-dimensional problem*

$$(-1)^m u^{(2m)} = f(u) \quad \text{in } (0, \infty), \qquad u(0) = u'(0) = \ldots = u^{(m-1)}(0) = 0$$

then $u \equiv 0$.

Proof. The differential equation admits a first integral given by

$$H = \sum_{i=1}^{m-1} (-1)^i u^{(i)} u^{(2m-i)} + (-1)^m \Big(\frac{1}{2} (u^{(m)})^2 + F(u) \Big),$$

where $F(s) := \int_0^s f(s)\, ds$. Indeed, we calculate that

$$\begin{aligned}
\frac{dH}{dt} &= \sum_{i=1}^{m-1} (-1)^i \Big(u^{(i+1)} u^{(2m-i)} + u^{(i)} u^{(2m-(i-1))} \Big) + (-1)^m \Big(u^{(m)} u^{(m+1)} + f(u) u' \Big) \\
&= -u' u^{(2m)} + (-1)^{m-1} u^{(m)} u^{(m+1)} + (-1)^m \Big(u^{(m)} u^{(m+1)} + f(u) u' \Big) \\
&= u' \Big((-1)^m f(u) - u^{(2m)} \Big) = 0 \quad \text{in } (0, \infty).
\end{aligned}$$

Suppose for contradiction that $u \not\equiv 0$. Since u has a Green function representation by Theorem 9, we infer that u is strictly positive in $(0, \infty)$, so that $u^{(2m)} = (-1)^{(m)} f(u)$ has no

zero in $(0, \infty)$. By the mean value theorem, this implies that $u^{(j)}$ has at most $2m - j$ zeros in $(0, \infty)$ for $j = 0, \dots, 2m$. Hence every $u^{(j)}$ is eventually monotone and has a limit as $t \to \infty$ since it is bounded. From this it clearly follows that

$$u^{(j)}(t) \to 0 \qquad \text{as } t \to \infty \text{ for } j = 1, \dots, 2m,$$

but then also $u(t) \to 0$ by using the equation and the assumptions on f again. Since $F(0) = 0$, we thus find that $H \equiv 0$ in $[0, \infty)$. In particular, the boundary conditions yield

$$0 = H(0) = \frac{(-1)^m}{2}[u^{(m)}(0)]^2,$$

so that $u^{(m)}(0) = 0$. If $m = 1$ we conclude $u(0) = u'(0) = 0$, so that $u \equiv 0$ by the uniqueness of the solution of the initial value problem – contrary to what we have assumed.

If $m > 1$, we set $v = -u''$ and find that v is a bounded solution of

$$(-1)^{m-1} v^{(2(m-1))} = f(u) > 0 \quad \text{in } (0, \infty), \qquad v(0) = v'(0) = \ldots = v^{(m-2)}(0) = 0.$$

So v also has a Green function representation by Theorem 9, and thus $v > 0$ in $(0, \infty)$. In sum, we have

$$u(0) = 0 \quad \text{and} \quad u > 0, u'' < 0 \quad \text{in } (0, \infty),$$

which forces $u'(0) > 0$ and therefore contradicts the boundary conditions. The proof is finished. $\square$

Proof of Theorem 4 (completed). By Corollary 13, we have $u(x) = \hat{u}(x_1)$ with some function $\hat{u} : [0, \infty) \to \mathbb{R}$. Since $\hat{u}$ satisfies the assumptions of Theorem 14, we conclude that $\hat{u} \equiv 0$ and therefore $u \equiv 0$. $\square$

5. Appendix

Lemma 15. *Let v be a classical bounded solution of $(-\Delta)^m v = g(v)$ in $\mathbb{R}^N$. If $g : \mathbb{R} \to [0, \infty)$ is convex and non-negative with $g(s) > 0$ for $s < 0$ then $v \geq 0$.*

Proof. Notice that the boundedness of v implies the boundedness of $D^j v$ for $j = 1, \dots, 2m$. First we show that $(-\Delta)^l v \geq 0$ in $\mathbb{R}^N$ for $l = 1, \dots, m-1$. Assume that there exists $l \in \{1, \dots, m-1\}$ and $x_0 \in \mathbb{R}^N$ with $(-\Delta)^l v(x_0) < 0$ but $(-\Delta)^j v \geq 0$ in $\mathbb{R}^N$ for $j = l+1, \dots, m$. We may assume w.l.o.g. that $x_0 = 0$. Let $v_l := (-\Delta)^l v$ for $l = 1, \dots, m-1$ and set $v_0 = v$. Then we have

$$-\Delta v_0 = v_1, \quad -\Delta v_1 = v_2, \quad \ldots \quad -\Delta v_{m-1} = g(v_0) \text{ in } \mathbb{R}^N.$$

If we define spherical averages $\bar{w}(x) = \frac{1}{r^{N-1}\omega_N} \oint_{\partial B_r(0)} w(y)\, d\sigma_y$, $r = |x|$ then the radial functions $\bar{v}_0, \bar{v}_1, \dots, \bar{v}_{m-1}$ satisfy

$$-\Delta \bar{v}_0 = \bar{v}_1, \quad -\Delta \bar{v}_1 = \bar{v}_2, \quad \ldots \quad -\Delta \bar{v}_{m-1} \geq g(\bar{v}_0) \text{ in } \mathbb{R}^N,$$

where we have used Jensen's inequality and the convexity of g. Since $v_l(0) < 0$ we also have $\bar{v}_l(0) < 0$. Moreover $(r^{N-1}\bar{v}_l')' = -r^{N-1}\bar{v}_{l+1}$ or $\leq -r^{N-1}g(\bar{v}_0)$, which in both cases is non-positive. Since $\bar{v}_l'(0) = 0$ we see that $\bar{v}_l'(r) \leq 0$ for all $r > 0$. In particular $\bar{v}_l(r) \leq \bar{v}_l(0) < 0$.

Next we integrate the equation

$$\Delta \bar{v}_{l-1} = -\bar{v}_l \geq -\bar{v}_l(0) > 0$$

and obtain $r^{N-1}\bar{v}'_{l-1}(r) \geq -\frac{r^N}{N}\bar{v}_l(0)$, i.e, $\bar{v}'_{l-1}(r) \geq -\frac{r}{N}\bar{v}_l(0)$. The unboundedness of $\bar{v}'_{l-1}$ yields a contradiction.

Finally, we need to show that $v = v_0 \geq 0$. Assume that $v_0(x_0) < 0$ and w.l.o.g. $x_0 = 0$. Since $\Delta\bar{v}_0 = -\bar{v}_1 \leq 0$ we see that $\bar{v}'_0(r) \leq 0$ and define $\alpha := \lim_{r\to\infty} \bar{v}_0(r) < 0$. Then $\lim_{r\to\infty} \Delta\bar{v}_{m-1}(r) = -g(\alpha) < 0$, i.e., $\Delta\bar{v}_{m-1}(r) \leq -\frac{1}{2}g(\alpha) < 0$ for $r \geq r_0$. By integration this leads to $\lim_{r\to\infty} \bar{v}'_{m-1}(r) = -\infty$, which contradicts the boundedness of $\bar{v}'_{m-1}$. This finishes the proof of the lemma. □

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